Network Formation with Heterogeneous Agents

Juan Larrosa and Fernando Tohmé
Departamento de Economía - Universidad Nacional del Sur
CONICET
Email: jlarrosa,ftohme@criba.edu.ar

Abstract

The present paper analyzes a network formation problem, mainly based on the framework presented in [Bala-Goyal 2000]. We depart from their assumptions in two crucial aspects. On one hand, we assume that connecting to an agent pays off not only for the number of connections that the agent can provide but also for her intrinsic value. Since the values of the agents (which represent the amounts of information held by the agents) differ from agent to agent, we are introducing heterogeneity in the framework. On the other hand, we assume that each path connecting two agents has an associated cost which is the sum of the number of edges it includes. We obtain as a result that the only Nash structure is the circle network, which emerges as a robust and optimal structure that maximizes the benefits of the interactions among agents while at the same time it minimizes the costs of network formation.

JEL Classification: D20, C72

1 Introduction

Interactions among different agents may be represented in several forms. A way of representing direct exchanges that has attracted a lot of interest in the last years is by means of *networks*. Because of its ease to grasp this tool of analysis was first adopted in sociology and anthropology. It constituted for the experts in those fields a pictorial way to understand the influence over individuals of their neighborhoods. Focusing on real social networks sociologists and anthropologists amassed a vast amount of evidence that helped to understand how human behavior is molded by the behavior of other agents.

In mathematical terms a network is a *graph*, where the nodes represent individual agents and the edge the links over which an utility good (e.g. information, personal prestige, etc.) is exchanged [Wasserman-Faust 1994]. The economic literature introduced recently game-theoretic tools in this framework. Instead of being concerned just with the descriptive aspects, some economic theorists tackled the issue of how networks arise, in the first place, and of what makes them stable or efficient [Jackson-Wolinsky 1996], [Bala-Goyal 2000],[Dutta-Jackson 2000]. The game-theoretic approach to networks exhibits two main approaches: one based in the notion of cooperative games and the other with a strong strategic flavor. The analysis based in cooperative games, as it is usual in that approach, focuses on the problem of the formation of coalitions among agents. The pervasive assumption of transferability of utilities makes the cooperative approach difficult to justify in many cases as well as computationally costly [Qin 1996], [Dutta et al. 1998], [Slikker-Nouweland 2001].

The strategic or non-cooperative approach, instead, just requires the definition of the strategies available to the agents as well as the characterization of the corresponding individual payoff functions. Agents decide whether or not to join the network, assessing the benefits of linking (or not) to other agents. The rational choices of the agents lead to Nash equilibria that support the networks that are the focus of analysis in this framework. This is, precisely, the approach we will adopt in this paper.

As said, a network can be seen as a graph. An important modelling decision is whether this graph is directed or non-directed. This choice of primitives has consequences also for the equilibrium outcomes in non-cooperative network games. Non-directed graphs are useful to represent situations in which the direction of flow of utility goods is less important or irrelevant [Dutta-Mutuswami 1997]. On the other hand, directed graphs reflect the importance of distinguishing which agent initiated the connection and of the direction of flow. A commonplace convention is to draw directed edges with arrowheads pointing towards the agent that decided to start a link [Bala-Goyal 2000], [Dutta-Jackson 2001].

In this paper we model networks as directed graphs with one-way flow. We call "information" (in a rather generic use of the term) to the utility good that flows in the networks. Each agent is endowed with some amount of information, but has a payoff function that depends positively on the amount of information to which she has access to. By establishing links to other agents she can obtain the information held by them, but se has to pay a small cost to establish those links. The problem is to determine which structures can arise as strategic equilibria in the interaction among the agents and whether they are optimal or not.

Instances of the network formation problem analyzed here can be observed in many areas. So, for example, consider the following scenario: suppose that Internet users are charged a constant small amount for every link they visit. When a user visits a web site, she has to pay that fee. If she follows a link she has to pay for the new connection but she has access to more information. If she follows a link in that page she accesses a different site where she can find more information, again at a small cost. The question is, which is the most efficient way to surf a series of web sites?

Closer to our framework, we may ask what kind of architecture for a local area network (LAN) increases the speed of flow while reducing losses. This is in fact analogous to our generic problem: a particular computer in the LAN may need to use resources from another computer in the network. It has to have an efficient protocol to choose to which machine to connect. At the same time it has to "pay" a small cost (in CPU time, say) to reach the one that allows it to get the highest amount of resources. Since this is true for all the machines in the LAN, its strategic outcome has to allow everyone to get to all the information available, paying as little as possible. As in our result (although for technological reasons) the final outcome is a circular network [Tanenbaum 1989].

This happens also in human organizations, as for example evaluation committees. They are usually constituted by experts in different fields. Each one has to rely on the others to obtain information out of her field of expertise. The circular network, in this case, minimizes the number of queries while at the same time it maximizes the information available to them. Some recent discussion about terrorist networks (e.g. Al-Qaeda) show that they share some of these features, namely that operative agents report and obeys to a boss, but they do not know other agents that also report to the same controller. The analogy with our framework resides in the behavior of the controllers: they demand obedience from plain operatives but do not provide them information. To establish a connection they have to pay for it (be it bribes, or just with compliments for their fanaticism). The architecture that may ensue in this case is not necessarily robust. It may break apart if just one of the n! odes i s being captured or killed [Arquilla-Ronfeldt 2001]. This case differs from the more general setting that will be analyzed in this paper. In our framework agents are not hierarchically ordered and therefore allows for circular networks as solutions.

In Section 2 we will begin our analysis with a presentation of the model. In Section 3 we will determine the equilibrium architecture and show how equilibria verify some criteria of stability and optimality. Finally, Section 4 ends the paper with a brief discussion of results reported here.

2 The Model

Let $N = (1, \ldots, n)$ be a set of agents. To avoid trivial results we will always assume that $n \geq 3$. If i and j are two typical members of N, a link among them, without intermediaries, originated by i and ending in j will be represented as ij. The interpretation of ij is that i establishes a contact with j that allows i to get acquainted with both the information possessed by j as well as her network of contacts. Each agent $i \in N$ has some information of her own, $I_i \in Z_+$, (i.e. represented as a nonnegative integer). As said i can have access to more information by forming links with other agents. The formation of links is costly, in time, resources and effort, but for simplicity we will assume that a link ij has a cost of 1 (in units of utility of information). By convention we assume that the information of each agent is worth the cost of establishing a link to her, i.e. that $I_i > 1$.

The agents will try to maximize the utility of the information available to them as well as to minimize the cost of connecting to other agents. In order to do this, they will be endowed with a set of *strategies*. Each strategy for $i \in N$ is a $(n-1)$-dimensional vector $g_i = \langle g_{i,1}, \ldots, g_{i,i-1}, g_{i,i+1} \ldots g_{i,n} \rangle$ where each $g_{i,j}$, for $j \neq i$, is either 0 or 1. This is interpreted as meaning that i establishes a direct link with j if $g_{i,j} = 1$ while if $g_{i,j} = 0$ there is no such direct link. The set of all i's strategies is denoted as G_i. Since we restrict our analysis to only pure strategies, $|G_i| = 2^{n-1}$. Finally, $G = G_1 \times \ldots \times G_n$ denotes the set of strategy profiles in the interaction among the agents in N.

The existence of a direct link ij indicates an asymmetric communication between i and j. That is, $g_{i,j} = 1$ indicates that i establishes a communication with j that permits i to access to j's information but no viceversa (the symmetry between i and j is restored if also $g_{j,i} = 1$). Structures with this feature are called **one-way flow** networks.

In one-way flow networks a strategy profile can be represented as a directed graph $g = (g_1 \ldots g_n)$ over N. That is, in the directed graph the elements of N are the *nodes* while any established link like $g_{i,j} = 1$ is represented by an arrow beginning in j with its head pointing to i.[1] That is, arrowheads always point toward the agent who establishes the link. It follows immediately that:

Proposition 1 *There exists a one-to-one map between directed graphs among n nodes and strategy profiles in G.*

Proof: A directed graph with n nodes is such that for each node i there exists at most one incoming arrow from each $j \neq i$ (and none from itself). Then, for each j define $g_{i,j}$ equal to 1 if there exists an incoming arrow from j, and 0 otherwise. This defines $g_i = \langle g_{i,1}, \ldots, g_{i,i-1}, g_{i,i+1}, \ldots, g_{i,n} \rangle$ for each $i \in N$. That is, it defines a $g = \langle g_1, \ldots, g_i, \ldots, g_n \rangle \in G$. Conversely, given g, a directed graph can be obtained by just adding an arrow from j to i if $g_{i,j} = 1$. Since $g_{i,i}$ is not defined, the graph is loop-less, and since $g_{i,j}$ has only two possible values, there exist either one or zero links between them. $\Box$

Example 1: consider a group of four agents, $N = \{a, b, c, d\}$. A joint strategy $g = \langle g_a, g_b, g_c, g_d \rangle$ can be represented as a table:

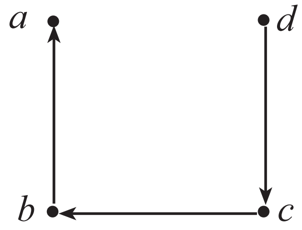
Strategy	**a**	**b**	**c**	**d**
g_a	X	1	0	0
g_b	0	X	1	0
g_c	0	0	X	1
g_d	0	0	0	X

Each row is the strategy chosen by one of the agents. Columns correspond to the agents. An entry 1 in row i and column j means that the strategy of agent i prescribes to establish a link with agent j. Entries in the diagonal are crossed out since agents cannot establish links with themselves. In Figure 1 we can see the directed graph that corresponds to g.

We define $N^{g_i} = \{k \in N | g_{i,k} = 1\}$ as the set of agents to whom i establishes a direct link according to her strategy g_i. We say that there exists a *path* from j

[1] In order to represent the idea that when i establishes a link with j, the information flows *from j to i*.

Figure 1:



to i according to $g \in G$ if there exists a sequence of different[2] agents $i, j_0 \ldots j_m, j$ such that $g_{i,j_0} = g_{j_0,j_1} = \ldots = g_{j_{m-1},j_m} = g_{j_m,j} = 1$. In other words, given the joint strategy g, we have that $j_0 \in N^{g_i}$, $j_1 \in N^{g_{j_0}}$, $\ldots$, $j \in N^{g_{j_m}}$. A path from j to i has a *length*, the cardinality of the sequence $j_0 j_1, j_1 j_2, \ldots, j_{m-1} j_m$, i.e. m, which indicates the number of intermediary links between j and i. Notice that a direct link is a path of length 1.

Example 1 revisited: Given the strategy $g = \langle g_a, g_b, g_c, g_d \rangle$, we have that $N^{g_a} = \{b\}$, $N^{g_b} = \{c\}$ and $N^{g_c} = \{d\}$ while $N^{g_d} = \emptyset$. This sequence establishes a path from d to a of length 3.

We denote the set of agents accessed (directly and otherwise) by i as $N^{i;g} = \{k \in N | k \rightarrow_g i\} \cup \{i\}$. We include i in $N^{i;g}$ to indicate that i knows her own valuation, despite the fact that we assumed that there is no direct link from i to herself. Let $\mu_i : G \rightarrow \{0, ..., n \times (n-1)\}$ be the number of links in all paths that end in i, originated by agents in $N^{i;g}$ under any given joint strategy.[3]

Example 2: Assume that we have $N = \{1, 2, 3, 4, 5\}$ and the strategy $g = \langle g_1, g_2, g_3, g_4, g_5 \rangle$ given by the following table:

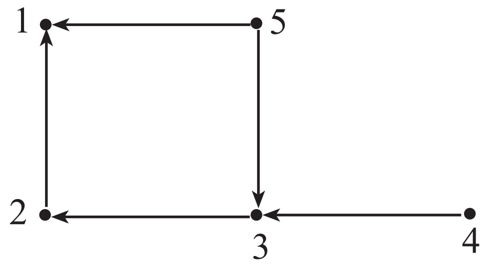
Strategy	**1**	**2**	**3**	**4**	**5**
g_1	X	1	0	0	1
g_2	0	X	1	0	0
g_3	0	0	X	1	1
g_4	0	0	0	X	0
g_5	0	0	0	0	X

Figure 2 shows the corresponding network. We have that $N^{1;g} = \{1, 2, 3, 4, 5\}$, $N^{2;g} = \{2, 3, 4, 5\}$, $N^{3;g} = \{3, 4, 5\}$ while $N^{4;g} = \{4\}$ and $N^{5;g} = \{5\}$. That is, under g we have that 1 can access to the information of all the agents while 4 and 5 have access only to their own information. The number of links required to obtain the information are $\mu_1(g) = 5$, $\mu_2(g) = 3$ and $\mu_3(g) = 2$, while $\mu_4(g) = \mu_5(g) = 0$.

[2]In order to avoid cycles.

[3]In a directed graph over n nodes there can be at most $n \times (n-1)$ direct links.

Figure 2:



To make this framework a game, we have to define the payoffs to the agents. Let $\Pi_i : G \to R$, the payoff function for agent i, be:

$$\Pi_i(g) \equiv \sum_{j \in N^{i;g}} I_j - \mu_i(g)$$

That is, i's payoff is just the sum of all the information that can be accessed by her, less the cost of the paths reaching her that are established according to g (recall that each link is assumed to have a unit cost). The intuition here is that i gets a payoff from accessing to more information but at the same time she has to pay a "fee" for each of the links on the paths to the sources of information.

Example 2 revisited: Suppose the information owned by the agents is: $I_1 = 2$, $I_2 = 2$, $I_3 = 4$, $I_4 = 3$ and $I_5 = 3$. Then, under strategy g we have that

$$\Pi_1(g) = I_1 + \cdots + I_5 - \mu_1(g) = 2 + 2 + 4 + 3 + 3 \ - \ 5 = 9$$

$$\Pi_2(g) = I_2 + \cdots + I_5 - \mu_2(g) = 2 + 4 + 3 + 3 \ - \ 3 = 9$$

$$\Pi_3(g) = I_3 + \cdots + I_5 - \mu_3(g) = 4 + 3 + 3 \ - \ 2 = 8$$

$$\Pi_4(g) = I_4 - \mu_4(g) = 3 \ - \ 0 = 3$$

$$\Pi_5(g) = I_5 - \mu_5(g) = 3 \ - \ 0 = 3$$

We can notice here that, for example, if $g_{1,5} = 0$, 1 could improve her payoff (i.e. obtaining 10 instead of 9) because she would still have access to I_5 but using one link less.

For each $g \in G$, agent i obtains a structure $N^{i;g}$ and her payoff depends critically on the type of directed graph that corresponds to $N^{i;g}$ as summarized in the following proposition:

Proposition 2 *Given two joint strategies g and $g^{'}$, $\Pi_i(g) \geq \Pi_i(g^{'})$ iff the corresponding graphs $N^{i;g}$ and $N^{i,g^{'}}$ are such that:*

$$\sum_{j \in N^{i;g}} I_j - \sum_{j \in N^{i;g^{'}}} I_j \geq \mu_i(g) - \mu_i(g^{'}).$$

Proof: *Trivial.* □

This result conveys the intuition that the goal of a rational agent is to get as much information as possible traversing as few links as possible. Two cases are of particular interest:

- $\sum_{j \in N^{i;g}} I_j = \sum_{j \in N^{i;g'}} I_j$ and $\mu_i(g) \leq \mu_i(g^{'})$.
- $\sum_{j \in N^{i;g}} I_j \geq \sum_{j \in N^{i;g'}} I_j$ and $\mu_i(g) = \mu_i(g^{'})$.

The first shows that $\Pi_i(g) \geq \Pi_i(g^{'})$ if the information obtained through g is the same as the one reached by means of $g^{'}$ but the number of links required is less in g than in $g^{'}$. The second case shows that $\Pi_i(g) \geq \Pi_i(g^{'})$ if the number of links required to reach the information is the same in g and $g^{'}$ but the amount of information obtained in g is more than the amount reached in $g^{'}$.

3 Equilibrium and Optimality

Given a network $g \in G$,[4] let g_{-i} be the directed graph obtained by removing all of agent i's direct links. Then, g can be written as $g = g_i \oplus g_{-i}$ where $\oplus$ indicates that g is formed by the union of the links of g_i and those in g_{-i}. A strategy g_i is said the *best response* of agent i to g_{-i} if

$$\Pi_i(g_i \oplus g_{-i}) \geq \Pi_i(g_i^{'} \oplus g_{-i})$$

for all $g_i^{'} \in G_i$

Example 3: *Consider again the case of $N = \{1, 2, 3, 4, 5\}$, where $I_1 = 2$, $I_2 = 2$, $I_3 = 4$, $I_4 = 3$ and $I_5 = 3$. Let g_{-1} be described by the following table:*

Strategy	**1**	**2**	**3**	**4**	**5**
g_2	0	X	1	0	1
g_3	0	0	X	1	0
g_4	0	0	0	X	0
g_5	0	0	0	0	X

See Figure 3 for the situation faced by 1.

She has to decide to whom establish a connection. A possibility is to remain isolated, but that would give her a payoff of only 2. Alternatively, she could connect to as many of the other agents as she likes. But some connections may be redundant in terms of the gain in information. Such redundancy, in turn, would mean a higher cost for the same information. So, for instance, to connect both to 3 and 4, would ensure 1 to have access to the information of 3 and 4. The number of links required would be 3. The payoff is then $2+4+3 - 3 = 6$. She could, instead, connect only to 3, since she would still get hold of the

[4] According to Proposition 1 we identify a joint strategy g with its corresponding directed graph.

Figure 3:

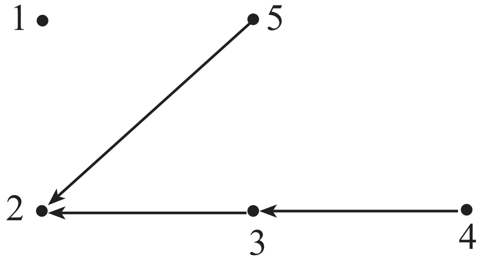
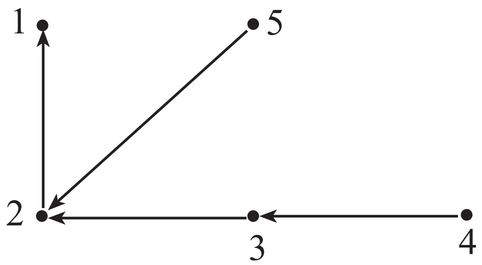


Figure 4:



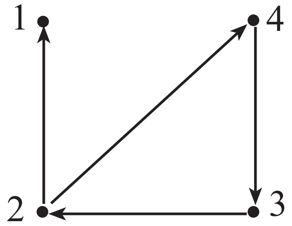
information of 3 and 4 but it would require only 2 links, i.e., her payoff would be $2+4+3\ -\ 2=7$. A bit of reflection shows that the best answer for 1 would be to connect only to the agent with the higher payoff under g_{-1}. That is, to agent 2, who has a payoff of $2+4+3+3\ -\ 3=9$. Then, 1 will reach the information of $2,3,4$ and 5, requiring 4 links. That is, her payoff would be of 10. Figure 5 shows the resulting network.

The set of best responses to g_{-i} is $BR_i(g_{-i})$. A network $g=\langle g_1,\ldots,g_n\rangle$ is said to be a *Nash network* if for each i, $g_i \in BR_i(g_{-i})$ i.e. if g (as a joint strategy) is a Nash equilibrium. In order to determine the structure of Nash networks let us give a few more definitions that will allow us to describe some additional properties of networks.

Given a network g, a set $C \subset N$ is called a *component* of g if for every pair of agents i and j in C $(i \neq j)$ we have that $j \in N^{i;g}$ and there does not exist $C^{'}$, $C \subset C^{'}$ for which this is true. A component C it said to be *minimal* if C is not a component anymore once a link $g_{i,j}=1$ between two agents i and j in C is cut off, i.e. if $g_{i,j}=0$.

Example 4:*If* $N=\{1,2,3,4\}$*, consider the following network* g*, represented in Figure* 5*:*

Figure 5:



Strategy	1	2	3	4
g_1	X	1	0	0
g_2	0	X	1	0
g_3	0	0	X	1
g_4	0	1	0	X

Clearly $C = \{2, 3, 4\}$ is a component, since $N^{2;g} = N^{3;g} = N^{4;g} = \{2, 3, 4\}$ and if we consider $N = C \cup \{1\}$, N is not a component, since 1 does not belong to $N^{2;g}$, $N^{3;g}$ or $N^{4;g}$. On the other hand, C is minimal, since if we cut off any of the links 23, 34 or 42 some of the agents are no longer reachable for at least one agent in C. So, for instance, if 23 is cut off, in the new network $g^{'}$ we have that $N^{2,g^{'}} = \{2\}$.

A network g it says to be *connected* if it supports a unique component. If that unique component is minimal, g it says to be *minimally connected.* A network that is not connected it says to be *disconnected.* A *circular network* is one in which the agents can be labelled (by means of a function $l : N \to N$) as $\{l(1), \ldots, l(n)\}$ and $g_{l(1),l(2)} = g_{l(2),l(3)} = \ldots = g_{l(n-1),l(n)} = g_{l(n),l(1)} = 1$ and there are no other links.

Then, with all these elements at hand we can state the following result:

Lemma 1 *A Nash network is circular.*

Proof: Let us consider $\Pi_i : G \to Z$, for each $i \in N$. If we show that there exists a unique (up to isomorphism) $g^* \in G$ that maximizes Π_i for each i, given that the others choose g^*_{-i} we would establish that there exists only one Nash equilibrium in the game. In fact, recalling Proposition 2 it is easy to show that for every i the corresponding payoff should be:

$$\Pi_i(g^*) = \sum_{i \in N} I_n - n$$

In words: the maximum of information that can be reached is the sum of all the information held by the agents while the minimum number of links that would allow to make that amount of information available to *all* of them is exactly n (i.e. a structure in which there exists only one path between any pair of agents).[5] It is easy to check that if all agents other than i choose g_j^*, i's choice will also be g_i^*. Suppose by way of contradiction that i has chosen $g_i^{'} \neq g_i^*$ such that $\Pi_i(g_i^{'}, g_i^*) > \Pi_i(g^*)$. That is, $\Pi_i(g_i^{'}, g_i^*) > \sum_{i \in N} I_n - n$. But since $\sum_{i \in N} I_n$ cannot be improved upon, the only possibility is that the number of links is smaller, i.e. $\Pi_i(g_i^{'}, g_i^*) = \sum_{i \in N} I_n - k$, where $k < n$. On the other hand, $k \not< (n-1)$ since otherwise i will not be able to access to at least one agent j and therefore cannot profit from her information I_j. Therefore, $\Pi_i(g_i^{'}, g_i^*) = \sum_{i \in N} I_n - (n-1)$. If so, one agent j will be accessed but will not have access to i's information because if every agent has access to I_i, and i has access to every I_j, $j \neq i$, the least number of links required is n.[6] But this is absurd, since each of the g_j^* prescribes to maximize Π_j and therefore, each j will have access to i and the number of links will be n.

As said, there may exist many structures g^* such that $\Pi_i(g^*) = \sum_{i \in N} I_n - n$. Since they have to support the maximum payoff for each agent, these g^*s have to include all the agents in N and must be connected (otherwise the information of the disconnected agents will be lost while the reduction in the cost of links will not be enough to compensate that loss [7]). On the other hand, any such g^* must be minimal since otherwise it would include a redundant link (which, if cut-off, leaves the structure connected) and we assume that g^* supports the optimal outcome for each agent, in which there are only n links and all the agents are connected. To show that a generic Nash network g^* is circular let us assume that it is not so. Therefore for every labelling $l : N \to N$ we have that either at least one of $g^*_{l(1),l(2)}, g^*_{l(2),l(3)}, \ldots, g^*_{l(n-1),l(n)}, g^*_{l(n),l(1)}$ has value 0 or there exists another link. This last possibility has to be discarded, since we know that g^* has only n links. Therefore it must not be possible to connect all the agents in N in such a way that each agent is connected to just one agent. But, since g^* has to include all the agents and connect them with only n links it is possible to choose one of the agents in the structure, say i, and attach to her a label, $l(i) = 1$. i is connected to just only one agent j since if i where connected to two different agents there would remain only $n-2$ links to connect the other $n-1$ agents. In that case at least one of the agents would not have a direct link pointed to her and therefore she would get a payoff far lower than

[5] In particular, this payoff is (weakly) better than isolation. Consider an agent i such that $I_i > I_j$ for every $j \neq i$. The payoff of remaining isolated is, for i, precisely I_i. Suppose by contradiction that $I_i + \sum_{j \neq i} I_j - n < I_i$. Then $\sum_{j \neq i} I_j < n$, but, as $I_j > 1$ for every j, $\sum_{j \neq i} I_j > n-1$, i.e. $\sum_{j \neq i} I_j \geq n$. Absurd.

[6] A simple proof ! by induction shows this. Assume that $N = \{i, j\}$. Then, if i has access to j and j to i, $|N|$ links are required. Assume that the claim is valid for $|N| = n-1$. If $|N| = n$ consider a subset $N^{'} \subset N$, $N = \{i\} \cup N^{'}$. Since $|N^{'}| = n-1$, all the elements in $N^{'}$ can be connected by $n-1$ links. To connect i to each element of $N^{'}$ in such a way that each agent in $N^{'}$ has access to i, just consider two agents in $N^{'}$, j_1 and j_2, such that the length of the path from j_1 to j_2 is $n-2$. Therefore, the length of the path from j_2 to j_1 is 1. Eliminate this single link and replace it by two links: one from j_2 to i and another from i to j_1. In this way, i will have access to all the agents in $N^{'}$ and each agent in $N^{'}$ will have access to i. The number of links is then $(n-1) - 1 + 2 = n$. Therefore, the claim is valid for every $n > 0$.

[7] Recall that each I_i is greater than the cost of a link.

the maximum. According to this, label the agent to whom i is connected, j, as $l(j) = 2$. Consider the only agent to whom j is connected, say k. Label k as $l(k) = 3$. Proceed in the same way until an agent is reached following the path of connections, say r, such that $l(r) = n$. Then, $n-1$ links would already have been used. It remains to establish to whom r is connected. It cannot be any of the agents labelled as $2, \ldots n-1$ since each of them, we argued has only one connection (towards the agent with the previous label). On the other hand, r cannot be connected to herself (otherwise her payoff would be only I_r). Therefore, she must be linked to i (who has label 1). That is, we found a labelling function l such that $g^*_{l(1)l(2)} = g^*_{l(2),l(3)} = \ldots = g^*_{l(n-1),l(n)} = g^*_{l(n),l(1)} = 1$. This contradicts our assumption that a Nash network is not circular. Therefore, g^* is circular.Finally, we claimed that there could exist many of these g^*s. The fact is that since they are circular, the only difference among them resides in the names of the agents. Therefore, two different Nash networks over N are isomorphic. That is, if g^* and $g^{*'}$ are two Nash networks over N, there exists a function $f : N \to N$ such that for each i and j we have that $g^*_{i,j} = g^{*'}_{f(i),f(j)}$. $\square$

To illustrate some of the claims in this argument consider the following example:

Example 5: *Let N be $\{1,2,3\}$ with $I_1 = 2$, $I_2 = 3$ and $I_3 = 4$. Let g^* be*

Strategy	**1**	**2**	**3**
g^*_1	X	1	0
g^*_2	0	X	1
g^*_3	1	0	X

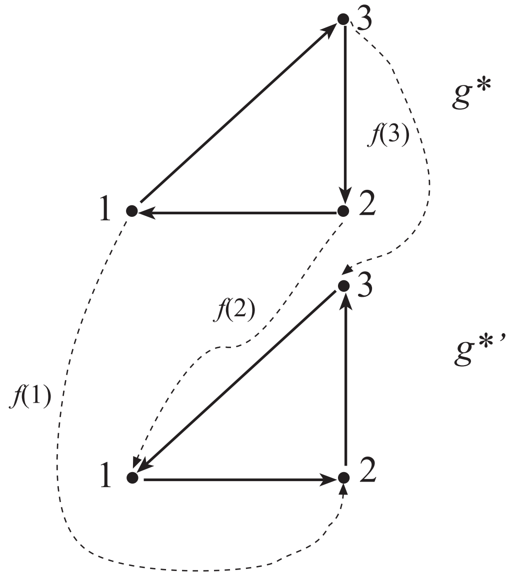
Let us check out that g^* is a Nash equilibrium. Consider the best response of 1 to g^*_{-1}. There are four options: $g^a_1 = \langle X,0,0\rangle$, $g^b_1 = \langle X,1,0\rangle$, $g^c_1 = \langle X,0,1\rangle$ or $g^d_1 = \langle X,1,1\rangle$. We have that $\Pi_1(g^a_1 \oplus g^*_1) = I_1 = 2$, $\Pi_1(g^b_1 \oplus g^*_1) = I_1 + I_2 + I_3 \ - \ 3 = 2+3+4 \ - \ 3 = 6$, $\Pi_1(g^c_1 \oplus g^*_1) = I_1 + I_3 \ - \ 2 = 2+4 \ - \ 2 = 4$ and $\Pi_1(g^d_1 \oplus g^*_1) = I_1 + I_2 + I_3 \ - \ 4 = 2+3+4 \ - \ 4 = 5$.[8] It is clear that g^b_1 is the best response to g^*_{-1}, but precisely $g^b_1 = g^*_1$. A similar argument is valid for g^*_2 and g^*_3. This shows that g^* is a Nash network. On the other hand, consider the following alternative network, $g^{*'}$ over the same N:

Strategy	**1**	**2**	**3**
$g^{*'}_1$	X	0	1
$g^{*'}_2$	1	X	0
$g^{*'}_3$	0	1	X

A brief examination shows that $g^{*'}$ is also a Nash network that for each agent in N yields the same payoff as g^*: $\Pi_1(g^{*'}) = \Pi_2(g^{*'}) = \Pi_3(g^{*'}) = I_1 + I_2 +$

[8]Notice, in the case of the network $g^c_1 \oplus g^*_1$, that $N^{1;g^c_1 \oplus g^*_1} = \{1,3\}$ and there are *two* paths that get to 1 from elements in this set: one that goes from 3 to 1 and the other that goes from 1, trough 3 , back to 1. This last path has two links, and includes the link corresponding to the other one.

Figure 6:



$I_3 - 3 = 2+3+4 - 3 = 6$. It is easy to establish an isomorphism $f : N \to N$ between g^* and $g^{*'}$: $f(1) = 2$, $f(2) = 1$ and $f(3) = 3$. Then, just consider the following table, obtained from the description of g^* by a transposition of rows and columns according to f:

Strategy	$f(1)$	$f(2)$	$f(3)$
$g^*_{f(1)}$	X	0	1
$g^*_{f(2)}$	1	X	0
$g^*_{f(3)}$	0	1	X

Notice that the structure of entries in this table is identical to the corresponding to $g^{*'}$. This establishes the isomorphism between g^* and $g^{*'}$. Figure 6 exhibits this isomorphism graphically.

According to this result, a stable outcome in the strategic interaction of agents is the circular network. We claim that it is stable because there is no incentives, once the circular structure arises, to cut-off links and form new ones, because the new configuration may at best achieve the same payoffs to the agents. This argument raises the question of the *optimality* of the outcome. That is, is there another configuration that may ensure better payoffs to the agents? Before answering negatively this question, let us introduce two different notions of optimality that may be worth to consider. One represents the notion of *social welfare* ensured by a network. Formally, let $W : G \to Z$ defined as

$W(g) = \sum_{i=1}^{n} \Pi_i(g)$ for $g \in G$. A network g is said *efficient* if $W(g) \geq W(g^{'})$ for all $g^{'} \in G$.

On the other hand, we have the notion of *Pareto optimality*. A network g is said Pareto optimal if there does not exist another network $g^{'}$ such that for each $i \in N$, $\Pi_i(g^{'}) \geq \Pi_i(g)$ and for at least one i, $\Pi_i(g^{'}) > \Pi_i(g)$.

We have then the following result:

Proposition 3 *A Nash network is both efficient and Pareto optimal.*

Proof: *Recall that a Nash network g^* supports the maximum payoff for each agent, $\Pi_i(g^*) = \sum_{i \in N} I_n - n$. Therefore, $\Pi_i(g^*) \geq \Pi_i(g)$ for each $i \in N$ and each $g \in G$. Therefore, g^* is Pareto optimal. By the same token, $W(g^*) = \sum_{i \in N} \Pi_i(g^*) \geq \sum_{i \in N} \Pi_i(g) = W(g)$ for each $g \in G$. That is, g^* is efficient.*$\square$

4 Discussion

We presented in this paper a model of network formation as a non-cooperative game where agents decide to whom to link by comparing the net benefits from their actions. The decisions are made simultaneously and therefore we do not require a dynamical setting which, instead, is present in [Bala-Goyal 2000]. Despite this, our framework does not differ that much from Bala and Goyal's and therefore our results are basically the same as those found dynamically. In a model where non-myopic agents have higher link formation costs than benefits a dynamic process converges again to a circular network [Watts 2002].

In our framework *heterogeneity* only means that each agent is endowed with some particular information that is valuable for the other agents. This assumption leads to an increase in the benefit from joining the network, making the participation in the network always more valuable than isolation. Consequently, the possibility of having an empty network as a Nash equilibrium (as in [Bala-Goyal 2000]) is dropped. On the other hand, it is interesting that circular network still arises as the main kind of equilibrium network.

References

[Arquilla-Ronfeldt 2001] Arquilla, J.- Ronfeldt, D. (eds.), *Networks and Netwars: The Future of Terror, Crime, and Militancy*, RAND MR-1382-OSD, 2001.

[Bala-Goyal 2000] Bala, V.- Goyal, S., A Noncooperative Model of Network Formation, *Econometrica* **68**:1181-1229, 2000.

[Dutta-Jackson 2000] Dutta, B. - Jackson, M., The Stability and Efficiency of Directed Communication Networks, *Review of Economic Design* **5**:251-272, 2000.

[Dutta-Jackson 2001] Dutta, B. - Jackson, M., On the Formation of Networks and Groups, in Dutta, B. and Jackson, M. (eds), *Models of the Strategic Formation of Networks and Groups*, Springer-Verlag, N.Y., 2001.

[Dutta et al. 1998] Dutta, B. - van den Nouweland, A. - Tijs, S., Link Formation in Cooperative Situations, *International Journal of Game Theory* **27**:245-256, 1998.

[Dutta-Mutuswami 1997] Dutta, B. - Mutuswami, S., Stable Networks, *Journal of Economic Theory* **76**:322-344, 1997.

[Jackson-Wolinsky 1996] Jackson, M. - Wolinsky, A., A Strategic Model of Social and Economic Networks, *Journal of Economic Theory* **71**:44-74, 1996.

[Qin 1996] Qin, C.Z., Endogenous Formation of Cooperation Structures, *Journal of Economic Theory* **69**:218-226, 1996.

[Slikker-Nouweland 2001] Slikker, M.- van den Nouweland, A., A One-Stage Model of Link Formation and Payoff Division, *Games and Economic Behavior* **34**:153-175, 2001.

[Tanenbaum 1989] Tanenbaum, A., *Computer Networks*, Prentice Hall, Englewood Cliffs (N.J.), 1989.

[Wasserman-Faust 1994] Wasserman, S.- Faust, K., *Social Networks Analysis*, Cambridge University Press, N.Y., 1994.

[Watts 2002] Watts, A., Non-myopic formation of circle networks, *Economics Letters* **74**:277-282, 2002.